Why Landauer's Formula for Resistance is Right

Economou and Soukoulis[1] have used the Kubo formula to calculate the electrical resistance of a disordered linear chain embedded in a perfect conductor. They find a resistance inversely proportional to $1/|t|^2$, where t is the amplitude transmission coefficient of the chain, in contradiction to Landauer's result[2] that the resistance is proportional to $1/|t|^2 - 1$. Fisher and Lee[3] have generalized the work of Economou and Soukoulis, and also get the same answer in conflict with Landauer. It is the purpose of this note to point out what these authors have done wrong, and show how a careful use of the Kubo formula leads to the Landauer result.

Economou and Soukoulis[1] consider the effect of a uniform ac field of frequency ω acting purely on the disordered part of the system. This should produce current pulses of wavelength v_F/ω traveling around the perfect conductor, and this wavelength has to be much less than the size of the system in order that $\hbar\omega$ should be much greater than the spacing between energy levels. These current pulses produce charge-density oscillations, and these should not be allowed. It is only reasonable to neglect interactions between electrons under conditions in which charge neutrality is maintained.

Although the electrons in the perfect conductor exhibit no resistance a field must be applied to accelerate them. This field must be $\pi/2$ out of phase with the field on the disordered section. In fact the system behaves like a resistor in series with a very large inductance. The inductance per unit length of the perfect conductor at zero temperature is $m/ne^2 = m\pi/e^2 k_F$. If ρ is the resistance per unit length of the disordered segment and $E\cos\omega t$ is the field driving the current through it, then a field $-(m\pi\omega E/e^2 k_F\rho)\sin\omega t$ has to be applied to the perfect conductor in order to maintain charge neutrality. Physically a neutralizing field is generated at a rate determined by the frequency of plasma oscillations. The matrix elements of this perturbing field therefore have the form

$$V_{\alpha\beta} = (eE/m\omega)\sin\omega t \int_0^l \psi_\alpha^*(x)\hat{p}\psi_\beta(x)dx - (\pi E/ek_F\rho)\cos\omega t \int_l^L \psi_\alpha^*(x)\hat{p}\psi_\beta(x)dx. \tag{1}$$

The disordered region extends from 0 to l, and the perfect conductor fills the rest of the cyclic system of length L. When evaluated between standing-wave states of the system with a partially reflecting barrier these matrix elements are

$$V_{\alpha\beta} = \pm(2iveEl/\omega L)[|t|\sin\omega t \pm (2\pi\hbar/e^2\rho)|r|\cos\omega t]. \tag{2}$$

From this point on the substitution in the Kubo formula proceeds as it does in Economou and Soukoulis, except for the presence of the extra term in Eq. (2). This leads to the modification of the equation $E^2/\rho l = E^2 e^2/2\pi\hbar$ by an extra term $2\pi\hbar E^2|r|^2/\rho^2 l^2 e^2$ on the right. This has the solution

$$\rho l = (2\pi\hbar/e^2)|r|^2/|t|^2, \tag{3}$$

which is the Landauer result. It satisfies the obvious condition that the resistance should go to zero when the length of the disordered region goes to zero, so that the reflection coefficient is zero. A detailed calculation of the current density shows that the space-dependent part is canceled and only the uniform diamagnetic term is left.

The method can be generalized to deal with the many-channel case considered by Fisher and Lee.[3] A recent preprint by Langreth and Abrahams[4] makes the same point. I do not think their method is identical with mine in the many-channel case. A detailed account of this method is being prepared.

D. J. Thouless

Department of Physics, University of Washington
Seattle, Washington 98195

[1]E. N. Economou and C. M. Soukoulis, Phys. Rev. Lett. 46, 618 (1981).
[2]R. Landauer, Philos. Mag. 21, 863 (1970).
[3]D. S. Fisher and P. A. Lee, to be published.
[4]D. C. Langreth and E. Abrahams, to be published.

Economou and Soukoulis Respond: In our paper[1] we have proved the following two statements: (a) If the disordered linear chain is connected to an incoherent current source I, then the resistance of the chain defined as V_0/I (where V_0 is voltage across the chain) equals $(\pi\hbar/e^2)(1/|t|^2 - 1)$ in agreement with Landauer's result.[2] On the other hand, (b) if the disordered chain is connected to a perfect conducting ring, and a uniform voltage V_0 is applied across the chain only, the current at the chain is $I = V_0/R$ where $R = (\pi\hbar/e^2)(1/|t|^2)$.

In his Comment[3] Thouless claims that the quantity $R = (\pi\hbar/e^2)(1/|t|^2)$ appearing in case (b) above should not be taken as the resistance of the chain because its operational definition produces an oscillating current in the perfect conductor, and such oscillations violating the local charge neutrality of the system should not be allowed. We think that there is no compelling physical reason for accepting this opinion. It is true that the accumulated charges would produce an induced field, which, however could be canceled by an addition-al external field. This would not change R, since the conductance is the response to the total field (external plus induced) and not simply to the external field. On the other hand, we recognize that the oscillating current in the perfect conductor for the configuration (b) is an undesirable feature.

The second point made by Thouless is that if one uses a modified configuration (b') including an additional uniform field V_1/L in the perfect conductor, such that the current uniformity is restored, then $R \equiv V_0/I$ is given again by $(\pi\hbar/e^2)(1/|t|^2 - 1)$ as in case (a). Such a result makes the case argued by Thouless very strong indeed.

By employing periodic boundary conditions we have calculated both the real and the imaginary parts of the conductivity, σ, of our system when connected to a perfect conducting ring. We found that the result for the case (b') is as claimed by Thouless. Here we summarize some important results of our calculation.

The conductivity $\sigma(x,x')$ is given in general by the following equation (in units where $e = m = \hbar = 1$):

$$\sigma(x,x') = \frac{i}{\omega}\left[n\delta(x-x') - \sum_{\alpha\beta}\frac{W_{\alpha\beta}(x)W_{\alpha\beta}{}^*(x')}{\omega_{\beta\alpha}}\right] + \frac{1}{2}\sum_{\alpha\beta}\frac{W_{\alpha\beta}(x)W_{\alpha\beta}{}^*(x')}{i\omega_{\beta\alpha}}\left(\frac{1}{\omega_{\beta\alpha}-\omega-i\epsilon} - \frac{1}{\omega_{\beta\alpha}+\omega+i\epsilon}\right), \quad (1)$$

where $\omega_{\beta\alpha} \equiv \omega_\beta - \omega_\alpha$, $H\psi_\alpha = \omega_\alpha\psi_\alpha$, $H\psi_\beta = \omega_\beta\psi_\beta$, $\omega_\beta > \omega_F$, $\omega_\alpha < \omega_F$, ω is the frequency of the field,

$$W_{\alpha\beta}(x) = \psi_\alpha{}^*(x)\partial\psi_\beta(x)/\partial x - \psi_\beta(x)\partial\psi_\alpha{}^*(x)/\partial x,$$

n is the electron density, and $\epsilon \to 0+$.

We found that for $|x|, |x'| \ll L$, the term in brackets in Eq. (1) is zero and that

$$\sigma(x,x') = \frac{1}{\pi}\left[\exp\left\{i\frac{\omega}{v_F}|x-x'|\right\} - (1-|t|^2)\exp\left\{i\frac{\omega}{v_F}(|x|+|x'|)\right\}\right]. \quad (2)$$

This result was first obtained by Lee.[4] Equation (2) is not appropriate for obtaining the response to a uniform field extending over the whole perfect conductor (because of the restrictions $|x|,|x'| \ll L$.) One must return to the general Eq. (1), and perform the integration over x' and then the summations over α and β. By doing so, we found that the integral of the term in brackets in Eq. (1) is not zero but it is equal to $n|t|^2$. The other term in Eq. (1) gives a result proportional to $|r|^2 \cong 1 - |t|^2$. So finally we obtain (for $|x| \ll L$)

$$\int_{-L/2}^{L/2}\sigma(x,x')dx' = \frac{i}{\omega}n|t|^2 - \frac{i}{\omega}n|r|^2\left\{\exp\left(i\frac{\omega}{v_F}|x|\right) - 1\right\} = \frac{i}{\omega}n - \frac{i}{\omega}n|r|^2\exp\left(i\frac{\omega}{v_F}|x|\right). \quad (3)$$

Taking into account Eqs. (2) and (3), and demanding that the current for $|x| \ll L$ must be uniform, we obtain for the configuration (b') Thouless's result. It is worthwhile to point out that if the term in brackets in Eq. (1) were identically zero for all x, x' then the result for the resistance would be $1/|t|^2$ and not $1/|t|^2 - 1$.

E. N. Economou
C. M. Soukoulis
  Corporate Research - Science Laboratories
  Exxon Research and Engineering Company
  Linden, N.J. 07036

Received 15 July 1981

[1]E. N. Economou and C. M. Soukoulis, Phys. Rev. Lett. **46**, 618 (1981).
[2]R. Landauer, Philos. Mag. **21**, 863 (1970).
[3]D. J. Thouless, preceding Comment.
[4]P. Lee, private communications.